Algorithmic complexity of the Lambek syntactic calculus

Mati Pentus

http://lpcs.math.msu.su/~pentus/

Moscow State University

- The **Lambek calculus** is a mathematical tool for formal language specification.
- Lambek grammars are equivalent to **context-free** grammars.
- The Lambek calculus provides a **complete** set of laws concerning formal language multiplication, division, and inclusion.
- The derivability problem for the Lambek calculus is decidable, but there is **no polynomial time deterministic algorithm** for deciding this problem (unless $\mathrm{P} = \mathrm{NP}$).

A *formal language* is a set of finite strings over a finite alphabet.

Example 1. Consider the alphabet $\Sigma = \{a, s, v\}$.
The set $\{vs, vsavs, vsavsavs, vsavsavsavs, \ldots\}$ is a formal language.

Two important approaches to formal language specification:

- Noam Chomsky (recursion-theoretic approach)
- Jim Lambek (logico-algebraic approach)
  J. Lambek, *The mathematics of sentence structure*, American Mathematical Monthly **65** (1958), no. 3, 154–170.

Syntactic types (parts of speech) are interpreted as formal languages, where the alphabet consists of natural language word-forms.
By $\circ$ we denote the concatenation operator
(example: $vs \circ avs = vsavs$).
Σ^* is the set of all strings over the alphabet Σ.
Σ^+ is the set of all non-empty strings over the alphabet Σ.

Definition 1. If $\mathcal{A} \subseteq \Sigma^*$ and $\mathcal{B} \subseteq \Sigma^*$, then

$$\mathcal{A} \cdot \mathcal{B} \rightleftharpoons \{x \circ y \mid x \in \mathcal{A},\ y \in \mathcal{B}\},$$
$$\mathcal{A} \backslash \mathcal{B} \rightleftharpoons \{y \in \Sigma^+ \mid \mathcal{A} \cdot \{y\} \subseteq \mathcal{B}\},$$
$$\mathcal{B} / \mathcal{A} \rightleftharpoons \{x \in \Sigma^+ \mid \{x\} \cdot \mathcal{A} \subseteq \mathcal{B}\}.$$

Example 2. $\{av, avs\} \cdot \{a, sa\} = \{ava, avsa, avssa\}$,
$\{av, avs\} \cdot \varnothing = \varnothing$.

Example 3. Let $\Sigma = \{\mathrm{John}, \mathrm{Mary}, \mathrm{runs}, \mathrm{student}, \mathrm{a}, \mathrm{the}\}$,
$L_n = \{\mathrm{student}\}$,
$L_{np} = \{\mathrm{John}, \mathrm{Mary}, \mathrm{a\ student}, \mathrm{the\ student}\}$,
$L_s = \{\mathrm{John\ runs}, \mathrm{Mary\ runs}, \mathrm{a\ student\ runs}, \mathrm{the\ student\ runs}\}$.
Then $L_{np} \backslash L_s = \{\mathrm{runs}\}$, $L_n \backslash L_s = \varnothing$, $L_{np}/L_n = \{\mathrm{a}, \mathrm{the}\}$,
$(L_{np}/L_n) \backslash L_s = \{\mathrm{student\ runs}\}$.

Here L_s contains all complete declarative sentences,
L_n contains all common nouns, and
L_{np} contains all proper names and all expressions which can occur in any context in which all proper names can occur.

Lemma 1. *If* $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \Sigma^*$, *then*
$(\mathcal{A} \cdot \mathcal{B}) \cdot \mathcal{C} = \mathcal{A} \cdot (\mathcal{B} \cdot \mathcal{C})$,
$\mathcal{A} \cdot (\mathcal{A} \backslash \mathcal{B}) \subseteq \mathcal{B}$,
$(\mathcal{A} \backslash \mathcal{B}) \cdot (\mathcal{B} \backslash \mathcal{C}) \subseteq \mathcal{A} \backslash \mathcal{C}$,
$\mathcal{A} \subseteq \mathcal{B} / (\mathcal{A} \backslash \mathcal{B})$.

J. Lambek proposed a deductive system that axiomatizes such laws.

Lemma 2. *Let* $\mathcal{A}, \mathcal{B} \subseteq \Sigma^*$. *Then the following conditions are equivalent:*

- $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{C}$,
- $\mathcal{A} \subseteq \mathcal{C} / \mathcal{B}$,
- $\mathcal{B} \subseteq \mathcal{A} \backslash \mathcal{C}$.

Definition 2. The set of all *types* is defined as the minimal set Tp such that

- $\{p_0, p_1, p_2, \ldots\} \subset \mathrm{Tp}$
- If $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, then $(A \cdot B) \in \mathrm{Tp}$, $(A \backslash B) \in \mathrm{Tp}$, and $(A/B) \in \mathrm{Tp}$.

Example 4. $(p_1 \cdot (p_1 \backslash p_2)) \in \mathrm{Tp}$.

Derivable objects of L_H are $A \to B$, where $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$.

Intuitively $\to$ is to be thought of as $\subseteq$.

Axioms and rules of $\mathsf{L_H}$ (the Lambek calculus)

$$A \to A \qquad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A \cdot B \to C}{A \to C / B} \qquad \frac{A \cdot B \to C}{B \to A \backslash C}$$

$$\frac{A \to C / B}{A \cdot B \to C} \qquad \frac{B \to A \backslash C}{A \cdot B \to C}$$

We write $\mathsf{L_H} \vdash A \to B$ for "$A \to B$ is derivable in the calculus $\mathsf{L_H}$".

Example 5. $\mathsf{L_H} \vdash p_1 \cdot (p_1 \backslash p_2) \to p_2$.

$$\frac{p_1 \backslash p_2 \to p_1 \backslash p_2}{p_1 \cdot (p_1 \backslash p_2) \to p_2}$$

Remark. $\mathsf{L_H} \nvdash p_2 \to p_1 \cdot (p_1 \backslash p_2)$, $\mathsf{L_H} \nvdash (p_1 \backslash p_2) \cdot p_1 \to p_2$.

Example 6. $A \cdot (B/C) \rightarrow (A \cdot B)/C$ is derivable in L_H.

$$\dfrac{\dfrac{(A \cdot (B/C)) \cdot C \rightarrow A \cdot ((B/C) \cdot C) \qquad \dfrac{\dfrac{\dfrac{B/C \rightarrow B/C}{(B/C) \cdot C \rightarrow B} \quad \dfrac{A \cdot B \rightarrow A \cdot B}{B \rightarrow A \backslash (A \cdot B)}}{(B/C) \cdot C \rightarrow A \backslash (A \cdot B)}}{A \cdot ((B/C) \cdot C) \rightarrow A \cdot B}}{(A \cdot (B/C)) \cdot C \rightarrow A \cdot B}}{A \cdot (B/C) \rightarrow (A \cdot B)/C}$$

Example 7.

$$\begin{array}{ll}
L_H \vdash A \cdot (A\backslash B) \rightarrow B, & \\
L_H \vdash (B/A) \cdot A \rightarrow B, & \\
L_H \vdash (A\backslash B) \cdot (B\backslash C) \rightarrow A\backslash C & \text{(Geach rule)}, \\
L_H \vdash (C/B) \cdot (B/A) \rightarrow C/A & \text{(Geach rule)}, \\
L_H \vdash A \rightarrow B/(A\backslash B) & \text{(Montague rule)}, \\
L_H \vdash A \rightarrow (B/A)\backslash B & \text{(Montague rule)}, \\
L_H \vdash (A\backslash B)/C \rightarrow A\backslash(B/C) & \text{(argument switching)}, \\
L_H \vdash A\backslash(B/C) \rightarrow (A\backslash B)/C & \text{(argument switching)}.
\end{array}$$

Definition 3. $A \underset{\mathsf{L_H}}{\leftrightarrow} B$ iff $\mathsf{L_H} \vdash A \to B$ and $\mathsf{L_H} \vdash B \to A$.

Example 8.

$$(A\backslash B)/C \underset{\mathsf{L_H}}{\leftrightarrow} A\backslash(B/C),$$

$$A/(B \cdot C) \underset{\mathsf{L_H}}{\leftrightarrow} (A/C)/B,$$

$$A \cdot (A\backslash(A \cdot B)) \underset{\mathsf{L_H}}{\leftrightarrow} A \cdot B.$$

$$A\backslash(A \cdot (A\backslash B)) \underset{\mathsf{L_H}}{\leftrightarrow} A\backslash B.$$

Example 9.

$$\mathsf{L_H} \vdash ((B/A)\backslash C)\backslash D \to (B\backslash C)\backslash(A\backslash D),$$

$$\mathsf{L_H} \nvdash ((A\backslash B)\backslash C)\backslash D \to C\backslash((B\backslash A)\backslash D).$$

Derivable objects of the calculus L (the Gentzen style Lambek calculus) are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^+$.

Axioms and rules of L

$$A \to A \qquad \frac{\Phi \to B \qquad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\mathrm{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash)\ (\Pi \text{ is non-empty}) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /)\ (\Pi \text{ is non-empty}) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \qquad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Here $A, B, C \in \mathrm{Tp}$ and $\Gamma, \Delta, \Phi, \Pi \in \mathrm{Tp}^*$.

Example 10. $\mathsf{L} \vdash A \cdot (B/C) \to (A \cdot B)/C$

$$\dfrac{A \to A \qquad \dfrac{C \to C \qquad B \to B}{(B/C)\, C \to B}\,(/\to)}{\dfrac{A\,(B/C)\, C \to (A \cdot B)}{\dfrac{A\,(B/C) \to (A \cdot B)/C}{A \cdot (B/C) \to (A \cdot B)/C}\,(\cdot\to)}\,(\to /)}\,(\to \cdot)$$

Theorem 1 (J. Lambek, 1958). $\mathsf{L} \vdash A_1 \ldots A_n \to B$ *if and only if* $\mathsf{L_H} \vdash A_1 \cdot \ldots \cdot A_n \to B$.

Theorem 2 (cut-elimination, J. Lambek, 1958). *A sequent is derivable in* L *if and only if it is derivable in* L *without* (cut).

Corollary. *The derivability problem for* L *(and for* $\mathsf{L_H}$*) is decidable.*

Remark. $\mathsf{L} \nvdash (A \cdot B)/C \to A \cdot (B/C)$.

Corollary (the subtype property). *If* $L \vdash \Gamma \to A$*, then there is a derivation of* $\Gamma \to A$ *such that all types appearing in this derivation are subtypes of those appearing in* $\Gamma \to A$*.*

The purpose of a Lambek categorial grammar is to provide an algorithm for distinguishing sentences from nonsentences in a fragment of a natural language.

Example 11. $\Sigma = \{\text{Mary}, \text{France}, \text{smiles}, \text{likes}\}$,

$$\begin{aligned} L_s = \{&\text{Mary smiles},\\ &\text{Mary likes Mary},\\ &\text{Mary likes France},\\ &\text{France smiles},\\ &\text{France likes Mary},\\ &\text{France likes France}\}. \end{aligned}$$

Definition 4. A *Lambek categorial grammar* is a triple $\langle \Sigma, D, f \rangle$ such that $|\Sigma| < \infty$, $D \in \mathrm{Tp}$, $f : \Sigma \to \mathcal{P}(\mathrm{Tp})$, and $|f(t)| < \infty$ for each $t \in \Sigma$.
The grammar *recognizes* the language

$$\mathcal{L}_L(\Sigma, D, f) \rightleftharpoons \{t_1 \ldots t_n \in \Sigma^+ \mid \exists B_1 \in f(t_1) \ldots \exists B_n \in f(t_n) \\ \mathsf{L} \vdash B_1 \ldots B_n \to D\}$$

Example 12.

$$np = p_1 \quad s = p_2 \quad D = s \quad \Sigma = \{\mathrm{Mary}, \mathrm{France}, \mathrm{smiles}, \mathrm{likes}\}$$

$$\begin{aligned} f(\mathrm{Mary}) = f(\mathrm{France}) &= \{np\} \\ f(\mathrm{smiles}) &= \{(np \backslash s)\} \\ f(\mathrm{likes}) &= \{((np \backslash s)/np)\} \end{aligned}$$

$$\dfrac{np \to np \qquad \dfrac{np \to np \qquad s \to s}{np\,(np\backslash s) \to s}\,(\backslash\to)}{\underset{\mathrm{Mary}}{np} \quad \underset{\mathrm{likes}}{((np\backslash s)/np)} \quad \underset{\mathrm{France}}{np} \quad \to s}\,(/\to)$$

B. Carpenter, *Type-Logical Semantics*, MIT Press, Cambridge, MA, 1997.
`http://www.colloquial.com/tlg/parser.html`

Example 13.

$$\dfrac{np \to np \qquad s \to s}{\underset{\text{Mary}}{np}\ \underset{\text{smiles}}{(np\backslash s)} \to s}\ (\backslash \to)$$

Example 14.

$$\dfrac{(np\backslash s) \to (np\backslash s) \qquad \dfrac{np \to np \qquad s \to s}{np\,(np\backslash s) \to s}\ (\backslash \to)}{\underset{\text{Mary}}{np}\ \underset{\text{smiles}}{(np\backslash s)}\ \underset{\text{charmingly}}{((np\backslash s)\backslash(np\backslash s))} \to s}\ (\backslash \to)$$

Example 15.

$$\Sigma = \{\text{Val}, \text{recommends}, \text{he}, \text{she}, \text{him}, \text{her}\}$$

$$\begin{aligned} f(\text{Val}) &= \{np\} \\ f(\text{recommends}) &= \{((np\backslash s)/np)\} \\ f(\text{he}) = f(\text{she}) &= \{(s/(np\backslash s))\} \\ f(\text{him}) = f(\text{her}) &= \{((s/np)\backslash s)\} \end{aligned}$$

$$\dfrac{\dfrac{\dfrac{np \longrightarrow np \qquad \dfrac{(np\backslash s) \longrightarrow (np\backslash s) \qquad s \longrightarrow s}{(s/(np\backslash s))\ (np\backslash s) \longrightarrow s}\,(/\longrightarrow)}{(s/(np\backslash s))\ ((np\backslash s)/np)\ np \longrightarrow s}\,(/\longrightarrow)}{(s/(np\backslash s))\ ((np\backslash s)/np) \longrightarrow (s/np)}\,(\longrightarrow/) \qquad s \longrightarrow s}{\underset{\text{She}}{(s/(np\backslash s))}\ \underset{\text{recommends}}{((np\backslash s)/np)}\ \underset{\text{him}}{((s/np)\backslash s)} \longrightarrow s}\,(\backslash\longrightarrow)$$

Example 16.

$$\Sigma = \{\text{John}, \text{Val}, \text{succeeds}, \text{exists}, \text{helps}, \text{recommends}, \text{student}, \text{professor}, \text{club}, \text{a}, \text{the}, \text{every}, \text{this}, \text{strange}, \text{whenever}, \text{whom}, \text{relatively}, \text{everywhere}, \text{or}\}$$

John succeeds whenever Val recommends a club or helps the student whom this relatively strange professor recommends.

$$\begin{aligned}
f(\text{John}) = f(\text{Val}) &= \{np\}\\
f(\text{succeeds}) = f(\text{exists}) &= \{(np\backslash s)\}\\
f(\text{helps}) = f(\text{recommends}) &= \{((np\backslash s)/np)\}\\
f(\text{student}) = f(\text{professor}) = f(\text{club}) &= \{n\}\\
f(\text{a}) = f(\text{the}) = f(\text{every}) &= \{(np/n)\}\\
f(\text{this}) &= \{(np/n), np\}\\
f(\text{strange}) &= \{(n/n)\}\\
f(\text{whenever}) &= \{((s\backslash s)/s)\}\\
f(\text{whom}) &= \{((n\backslash n)/(s/np))\}\\
f(\text{relatively}) &= \{((n/n)/(n/n))\}\\
f(\text{everywhere}) &= \{((np\backslash s)\backslash(np\backslash s))\}
\end{aligned}$$

$$f(\text{or}) = \{((np\backslash np)/np), ((s\backslash s)/s), (((np\backslash s)\backslash(np\backslash s))/(np\backslash s))\}$$

Example 17.
John eats rice quickly and beans slowly.
John gave a book to Bill and a record to Max.
Australia exports and Europe imports butter.

Theorem 3 (1992). *Let $\mathcal{A} \subseteq \Sigma^+$. The language $\mathcal{A}$ is recognized by some Lambek categorial grammar if and only if $\mathcal{A}$ is context-free.*

Definition 5. A *language model* (*free semigroup model*) is a pair $\langle \Sigma^+, v \rangle$ such that Σ is a finite or countable alphabet and

- $v(p_i) \subseteq \Sigma^+$,
- $v(A \cdot B) = v(A) \cdot v(B)$,
- $v(A \backslash B) = v(A) \backslash v(B) = \{y \in \Sigma^+ \mid v(A) \cdot \{y\} \subseteq v(B)\}$,
- $v(B / A) = v(B) / v(A) = \{x \in \Sigma^+ \mid \{x\} \cdot v(A) \subseteq v(B)\}$.

A sequent $A \to B$ is *true in the model* $\langle \Sigma^+, v \rangle$ iff $v(A) \subseteq v(B)$.

Example 18. Let $p, q \in \{p_0, p_1, p_2, \ldots\}$. Consider the language model where $\Sigma = \{a_1, a_2\}$, $v(p) = \{a_1\}$, and $v(q) = \{a_2\}$. Then

$$v(q \backslash q) = \varnothing,$$
$$v(p \cdot (q \backslash q)) = \varnothing,$$

$$v(p) = \{a_1\} \not\subseteq \varnothing = v(p \cdot (q \backslash q)).$$

Thus $p \to p \cdot (q \backslash q)$ is false in this model. Note that $\mathrm{L} \nvdash p \to p \cdot (q \backslash q)$.

Example 19. Let $p, q, r \in \{p_0, p_1, p_2, \ldots\}$.

$$\Sigma = \{a_1, a_2, a_3\} \qquad \begin{aligned} v(p) &= \{a_1 a_2\} \\ v(q) &= \{a_3\} \\ v(r) &= \{a_2 a_3\} \end{aligned}$$

$$\begin{aligned} v(p \cdot q) &= \{a_1 a_2 a_3\} \\ v((p \cdot q)/r) &= \{a_1\} \\ v(q/r) &= \varnothing \\ v(p \cdot (q/r)) &= \varnothing \end{aligned}$$

$$v((p \cdot q)/r) = \{a_1\} \not\subseteq \varnothing = v(p \cdot (q/r))$$

Thus $(p \cdot q)/r \to p \cdot (q/r)$ is false in this model. Note that $\mathrm{L} \nvdash (p \cdot q)/r \to p \cdot (q/r)$.

Remark. L is sound with respect to language models (i. e., every derivable sequent is true in every language model).

Theorem 4 (completeness, 1993). *A sequent is derivable in* L *if and only if it is true in every language model.*

Derivable objects of the calculus L^* are *sequents* $\Gamma \to A$, where $A \in \mathrm{Tp}$ and $\Gamma \in \mathrm{Tp}^*$.

Axioms and rules of L^*

$$A \to A \qquad \frac{\Phi \to B \qquad \Gamma\, B\, \Delta \to A}{\Gamma\, \Phi\, \Delta \to A}\ (\mathrm{cut})$$

$$\frac{A\, \Pi \to B}{\Pi \to A \backslash B}\ (\to \backslash) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, \Phi\, (A \backslash B)\, \Delta \to C}\ (\backslash \to)$$

$$\frac{\Pi\, A \to B}{\Pi \to B / A}\ (\to /) \qquad \frac{\Phi \to A \qquad \Gamma\, B\, \Delta \to C}{\Gamma\, (B / A)\, \Phi\, \Delta \to C}\ (/ \to)$$

$$\frac{\Gamma \to A \qquad \Delta \to B}{\Gamma\, \Delta \to A \cdot B}\ (\to \cdot) \qquad \frac{\Gamma\, A\, B\, \Delta \to C}{\Gamma\, (A \cdot B)\, \Delta \to C}\ (\cdot \to)$$

Example 20.

$$\frac{A \to A \qquad \dfrac{B \to B}{\to B \backslash B}\ (\to \backslash)}{A \to A \cdot (B \backslash B)}\ (\to \cdot)$$

Remark. $L^* \vdash A \to A \cdot (B \backslash B)$, but $L \nvdash A \to A \cdot (B \backslash B)$.

Cut-elimination theorem. *We may drop* (cut).

Noncommutative linear logic was suggested by J.-Y. Girard in 1987 and expounded by D. N. Yetter.

D. N. Yetter, *Quantales and noncommutative linear logic*, Journal of Symbolic Logic, **55** (1990), no. 1, pp. 41–64.

Definition 6. Let $\mathrm{At} \rightleftharpoons \{p_0, p_1, p_2, \ldots\} \cup \{\overline{p_0}, \overline{p_1}, \overline{p_2}, \ldots\}$. *Linear formulas* are the elements of the minimal set Fm such that

- $\mathrm{At} \subset \mathrm{Fm}$,
- if $A \in \mathrm{Fm}$ and $B \in \mathrm{Fm}$, then $(A \otimes B) \in \mathrm{Fm}$ and $(A \invamp B) \in \mathrm{Fm}$.

$$(p_i)^\perp \rightleftharpoons \overline{p_i} \qquad (\overline{p_i})^\perp \rightleftharpoons p_i$$

$$(A \otimes B)^\perp \rightleftharpoons (B)^\perp \invamp (A)^\perp \qquad (A \invamp B)^\perp \rightleftharpoons (B)^\perp \otimes (A)^\perp$$

Example 21.
$((\overline{p} \invamp ((\overline{r} \invamp (\overline{r} \otimes r)) \otimes r)) \otimes q)^\perp = (\overline{q} \invamp ((\overline{r} \invamp ((\overline{r} \invamp r) \otimes r)) \otimes p))$.

Definition 7. The following function $\tau\colon \mathrm{Tp} \to \mathrm{Fm}$ embeds L^* into cyclic linear logic.

$$\begin{aligned}
\tau(p_i) &\rightleftharpoons p_i \\
\tau(A \cdot B) &\rightleftharpoons \tau(A) \otimes \tau(B) \\
\tau(A \backslash B) &\rightleftharpoons \tau(A)^{\perp} \parr \tau(B) \\
\tau(A / B) &\rightleftharpoons \tau(A) \parr \tau(B)^{\perp}
\end{aligned}$$

Example 22. $\tau(p_1/(p_2 \cdot p_3)) = p_1 \parr (\overline{p_3} \parr \overline{p_2})$

Derivable objects of cyclic linear logic are *sequents*

$$\to A_1 \ \ldots \ A_n,$$

where $A_i \in \mathrm{Fm}$.
The intended meaning of $\to A_1 \ \ldots \ A_n$ is $A_1 \parr \ldots \parr A_n$.

Axioms and rules

$$\to A^\perp\, A \qquad \frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta}\ (\parr) \qquad \frac{\to \Gamma\, A \qquad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta}\ (\otimes)$$

$$\frac{\to \Gamma\, \Delta}{\to \Delta\, \Gamma}\ (\text{rotate}) \qquad \frac{\to \Gamma\, A \qquad \to A^\perp\, \Delta}{\to \Gamma\, \Delta}\ (\text{cut})$$

Cut-elimination theorem. *We may drop* (cut).

Another calculus for the same logic.
Axioms and rules of MCLL

$$\to \overline{p_i}\, p_i \qquad \to p_i\, \overline{p_i}$$

$$\frac{\to \Gamma\, A\, B\, \Delta}{\to \Gamma\, (A \parr B)\, \Delta} \qquad \frac{\to \Gamma\, A \qquad \to \Phi\, B\, \Delta}{\to \Phi\, \Gamma\, (A \otimes B)\, \Delta} \qquad \frac{\to \Gamma\, A\, \Pi \qquad \to B\, \Delta}{\to \Gamma\, (A \otimes B)\, \Delta\, \Pi}$$

Example 23. $\mathrm{MCLL} \vdash\ \to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \parr p)$.

$$\dfrac{\dfrac{\dfrac{\to \overline{p}\,p \qquad \to q\,\overline{q}}{\to (\overline{p} \otimes q)\,\overline{q}\,p} \qquad \to r\,\overline{r}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,\overline{r}\,p}}{\to (\overline{p} \otimes q)\,(\overline{q} \otimes r)\,(\overline{r} \parr p)}$$

Example 24. $\mathrm{MCLL} \vdash\ \to (\overline{r} \otimes r)\,(\overline{r} \otimes r)\,(\overline{r} \parr r)$

Remark. $\mathrm{L}^* \vdash A_1 \ \ldots\ A_n \to B$ if and only if $\mathrm{MCLL} \vdash\ \to \tau(A_n)^{\perp}\ \ldots\ \tau(A_1)^{\perp}\,\tau(B)$.

Example 25. $\mathrm{L}^* \vdash ((q \backslash r) \cdot s) \to (q \backslash (r \cdot s))$ and $\mathrm{MCLL} \vdash\ \to (\overline{s} \parr (\overline{r} \otimes q))\,(\overline{q} \parr (r \otimes s))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\to \overline{r}\,r \qquad \to \overline{s}\,s}{\to \overline{s}\,\overline{r}\,(r \otimes s)} \qquad \to q\,\overline{q}}{\to \overline{s}\,(\overline{r} \otimes q)\,\overline{q}\,(r \otimes s)}}{\to \overline{s}\,(\overline{r} \otimes q)\,(\overline{q} \parr (r \otimes s))}}{\to (\overline{s} \parr (\overline{r} \otimes q))\,(\overline{q} \parr (r \otimes s))}}{}$$

M. Pentus, *Lambek calculus is* NP-*complete*, CUNY Ph.D. Program in Computer Science Technical Report TR-2003005, CUNY Graduate Center, New York, May 2003.
`http://www.cs.gc.cuny.edu/tr/techreport.php?id=79`

Remark. The derivability problem for MCLL is in NP.

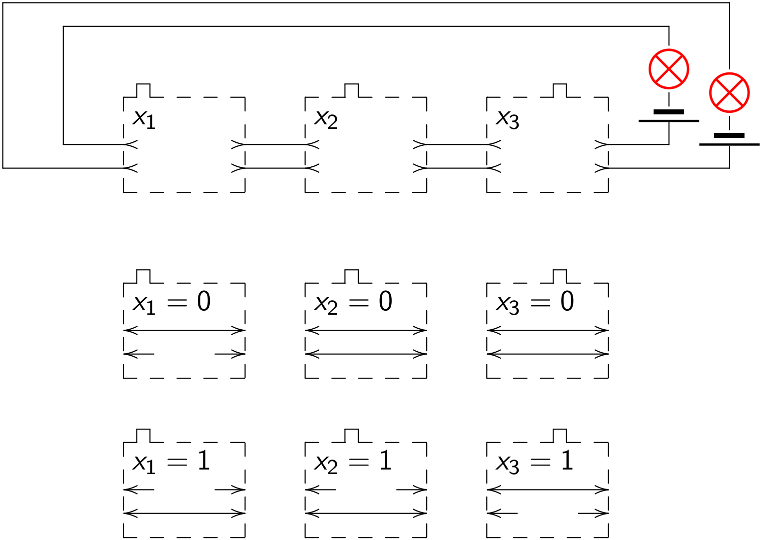
Theorem 5 (2003). *The derivability problem for* MCLL *is* NP-*hard.*

Corollary. *The derivability problem for* MCLL *is* NP-*complete.*

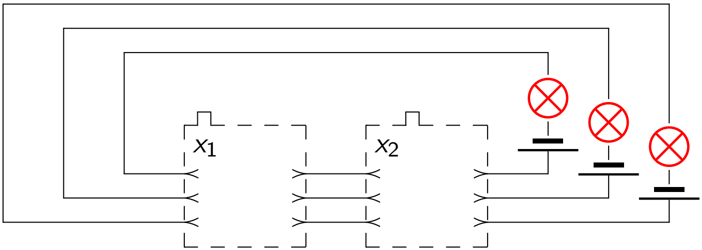
We shall reformulate the well-known NP-complete problem *SAT* (satisfiability in the classical propositional logic) in terms of electrical circuits. This makes it easy to translate the problem into a problem concerning planar graphs.

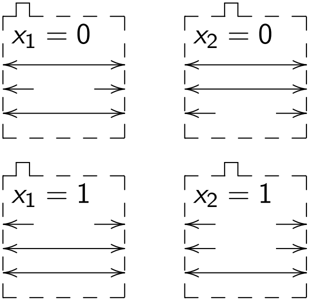
Let $c_1 \wedge \ldots \wedge c_m$ be a Boolean formula in conjunctive normal form with clauses $c_1, \ldots, c_m$ and variables $x_1, \ldots, x_n$.
We construct a frame (with m lamps and n sockets) and a set of $2n$ blocks (each of which fits into one socket only) so that the formula $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if there is a way to plug n blocks into the sockets so that no lamp will be switched on. Each block (and each socket) has $2m$ contacts.

Example 26. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$.



Example 27. $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2)$.





To model the circuits in MCLL we shall construct formulas G, $E_i(0)$, $E_i(1)$, F_i (where $1 \leq i \leq n$) such that

- $c_1 \wedge \ldots \wedge c_m$ is satisfiable if and only if MCLL $\vdash\ \rightarrow E_1(t_1)\ \ldots\ E_n(t_n)\ G$ for some $t_1, \ldots, t_n \in \{0, 1\}$,
- MCLL $\vdash\ \rightarrow F_1\ \ldots\ F_n\ G$ if and only if MCLL $\vdash\ \rightarrow E_1(t_1)\ \ldots\ E_n(t_n)\ G$ for some $t_1, \ldots, t_n \in \{0, 1\}$.

This can be done in polynomial time (in terms of the length of the given Boolean formula).

We shall denote p_{n+1} by r.
In the following definitions $1 \le j < m$, $1 \le i \le n$ and $t \in \{0, 1\}$.

$$
\begin{aligned}
G^0 &\rightleftharpoons (\overline{r} \parr r),\\
G^j &\rightleftharpoons ((\overline{r} \parr G^{j-1}) \otimes r),\\
G &\rightleftharpoons ((\overline{p_n} \parr G^{m-1}) \otimes p_0),\\
H^0 &\rightleftharpoons (\overline{r} \otimes r),\\
H^j &\rightleftharpoons ((\overline{r} \parr H^{j-1}) \otimes r),\\
H_i &\rightleftharpoons ((\overline{p_{i-1}} \parr H^{m-1}) \otimes p_i),\\
E_i^0(t) &\rightleftharpoons (\overline{r} \otimes r),\\
E_i^j(t) &\rightleftharpoons \begin{cases} (\overline{r} \parr (E_i^{j-1}(t) \otimes r)) & \text{if } [\![x_i]\!] = t \rightarrow [\![c_j]\!] = 1,\\ ((\overline{r} \parr E_i^{j-1}(t)) \otimes r) & \text{otherwise,} \end{cases}\\
E_i(t) &\rightleftharpoons \begin{cases} (\overline{p_{i-1}} \parr (E_i^{m-1}(t) \otimes p_i)) & \text{if } [\![x_i]\!] = t \rightarrow [\![c_m]\!] = 1,\\ ((\overline{p_{i-1}} \parr E_i^{m-1}(t)) \otimes p_i) & \text{otherwise,} \end{cases}\\
F_i &\rightleftharpoons ((E_i(0) \otimes H_i^{\perp}) \parr H_i \parr (H_i^{\perp} \otimes E_i(1))).
\end{aligned}
$$

Lemma 3. $\mathrm{MCLL} \vdash \ \rightarrow E_i(t)\, H_i^{\perp}$ *for each* $1 \leq i \leq n$ *and* $t \in \{0, 1\}$.

Lemma 4. $\mathrm{MCLL} \vdash \ \rightarrow F_i\, E_i(t)^{\perp}$ *for each* $1 \leq i \leq n$ *and* $t \in \{0, 1\}$.

Lemma 5. *If* $\mathrm{MCLL} \vdash \ \rightarrow \Gamma\, A^{\perp}$ *and* $\mathrm{MCLL} \vdash \ \rightarrow \Phi\, A\, \Delta$, *then* $\mathrm{MCLL} \vdash \ \rightarrow \Phi\, \Gamma\, \Delta$.

Theorem 6 (2003). *The derivability problems for* L^* *and* L *are* NP*-complete.*

Corollary. *The general membership problem for Lambek categorial grammars (where the input consists of a grammar and a string) is* NP*-complete.*

Definition 8. $L(\backslash, /)$ is the fragment of L without $\cdot$ (only product-free types are allowed).

Remark. L is conservative over $L(\backslash, /)$, i. e., if a product-free sequent is deriavble in L then it is derivable in $L(\backslash, /)$.

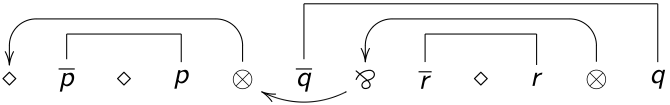
Remark. L^* is conservative over $L(\backslash, /)^*$.

Remark. It is unknown whether the derivability problems for $L(\backslash, /)^*$ and $L(\backslash, /)$ are NP-complete.

Example 28. The derivation

$$\dfrac{\to \overline{p}\; p \qquad \dfrac{\to \overline{r}\; r \qquad \to \overline{q}\; q}{\to \overline{q}\; \overline{r}\,(r \otimes q)}}{\to \overline{p}\,(p \otimes (\overline{q} \parr \overline{r}))\,(r \otimes q)}$$

corresponds to the following proof net.



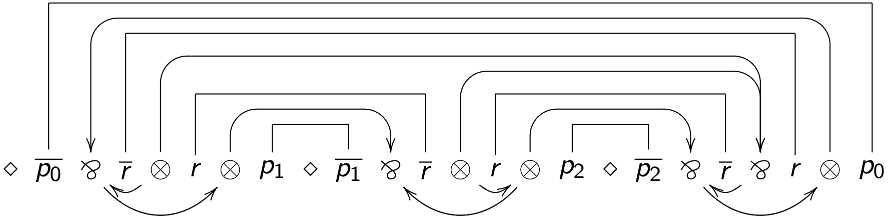
A *proof net* for Γ must satisfy the following conditions.

- $|\Gamma|_{\parr} + |\Gamma|_{\diamond} = |\Gamma|_{\otimes} + 2$.
- No intersections.
- Acyclic.

Example 29. Let

$$\Gamma = ((\overline{p_0} \parr (\overline{r} \otimes r)) \otimes p_1)\,(\overline{p_1} \parr ((\overline{r} \otimes r) \otimes p_2))\,((\overline{p_2} \parr (\overline{r} \parr r)) \otimes p_0).$$

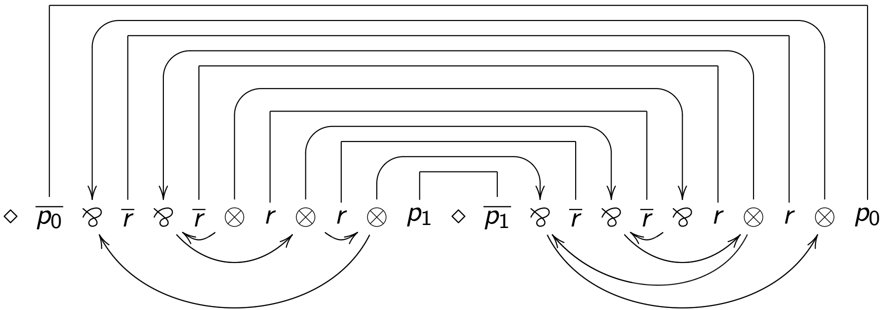
The following figure shows a proof net for Γ.



Example 30. Let

$$\Gamma = (\overline{p_0} \parr (((\bar{r} \parr (\bar{r} \otimes r)) \otimes r) \otimes p_1)) \, ((\overline{p_1} \parr ((\bar{r} \parr (\bar{r} \parr r)) \otimes r)) \otimes p_0).$$

The following is not a valid proof net for Γ (it contains a cycle).



Definition 9. $\|\cdot\| \colon \mathrm{Fm} \to \mathbb{Z}$

$$\begin{aligned}
\|p_i\| &= \|\overline{p_i}\| \rightleftharpoons 2,\\
\|A \otimes B\| &= \|A \parr B\| \rightleftharpoons \|A\| + \|B\|,\\
\|A_1 \ldots A_n\| &\rightleftharpoons \|A_1\| + \ldots + \|A_n\|.
\end{aligned}$$

Definition 10. $\mathrm{Occ} \rightleftharpoons \mathrm{Fm} \times \mathbb{Z}$.

Definition 11. $c \colon \mathrm{Occ} \to \mathbb{Z}$

$$\begin{aligned}
c(p_i) &= c(\overline{p_i}) \rightleftharpoons 1,\\
c(A \otimes B) &= c(A \parr B) \rightleftharpoons \|A\|.
\end{aligned}$$

Definition 12. $\prec$ is the following binary relation on Occ.

$$\begin{aligned}
&\langle A, k - \|A\| + c(A)\rangle \prec \langle (A \,\lambda\, B), k\rangle,\\
&\langle B, k + c(B)\rangle \prec \langle (A \,\lambda\, B), k\rangle,\\
&\text{if } \langle A, i\rangle \prec \langle B, j\rangle \text{ and } \langle B, j\rangle \prec \langle C, k\rangle, \text{ then } \langle A, i\rangle \prec \langle C, k\rangle.
\end{aligned}$$

Here $\lambda \in \{\otimes, \parr\}$.

Definition 13. Let $\diamond \notin \mathrm{Fm}$. Let $\Gamma = A_1 \dots A_n$. Then $\boldsymbol{\Omega}_\Gamma \rightleftharpoons \langle \Omega_\Gamma, \prec_\Gamma, <_\Gamma \rangle$, where

$$\Omega_\Gamma \rightleftharpoons \{\langle B, k + \|A_1 \dots A_{i-1}\| \rangle \mid 1 \le i \le n \text{ and } \langle B, k \rangle \preceq \langle A_i, c(A_i) \rangle\} \cup \{\langle \diamond, \|A_1 \dots A_{i-1}\| \rangle \mid 1 \le i \le n\},$$

$\langle A, k \rangle \prec_\Gamma \langle B, l \rangle$ iff $A \neq \diamond$, $B \neq \diamond$, and $\langle A, k \rangle \prec_\Gamma \langle B, l \rangle$,

$\langle A, k \rangle <_\Gamma \langle B, l \rangle$ iff $k < l$.

Definition 14.

$$\Omega_\Gamma^\diamond \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = \diamond\},$$
$$\Omega_\Gamma^{\mathrm{At}} \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C \in \mathrm{At}\},$$
$$\Omega_\Gamma^\otimes \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = A \otimes B \text{ for some } A \text{ and } B\},$$
$$\Omega_\Gamma^\invamp \rightleftharpoons \{\langle C, k \rangle \in \Omega_\Gamma \mid C = A \invamp B \text{ for some } A \text{ and } B\}.$$

Definition 15. A *proof net for* Γ is a relational structure $\langle \Omega_\Gamma, \mathcal{A}, \mathcal{E} \rangle$, where

- $\flat(\Omega_\Gamma^{\invamp}) + \flat(\Omega_\Gamma^{\diamond}) - \flat(\Omega_\Gamma^{\otimes}) = 2$,
- $\mathcal{A}$ is a map from $\Omega_\Gamma^{\otimes}$ to $\Omega_\Gamma^{\invamp} \cup \Omega_\Gamma^{\diamond}$,
- $\mathcal{E}$ is a map from $\Omega_\Gamma^{\mathsf{At}}$ to $\Omega_\Gamma^{\mathsf{At}}$,
- if $\langle \alpha, \beta \rangle \in \mathcal{E}$, then $\langle \beta, \alpha \rangle \in \mathcal{E}$,
- if $\langle \langle A, i \rangle, \langle B, j \rangle \rangle \in \mathcal{E}$, then $A = B^\perp$,
- the edges of the graph $\langle \Omega_\Gamma, \mathcal{A} \cup \mathcal{E} \rangle$ can be drawn without intersections on a semiplane while the vertices of the graph are ordered according to $<_\Gamma$ on the border of the semiplane,
- the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A} \rangle$ is acyclic.

Theorem 7 (1998). MCLL $\vdash\ \to \Gamma$ *if and only if there exists a proof net for* Γ.